

# FIRST THEOREMS OF PROPOSITIONAL CALCULUS

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ABSTRACT. This module includes first proofs of propositional calculus theorems. The following theorems and proofs are adapted from D. Hilbert and W. Ackermann's 'Grundzuege der theoretischen Logik' (Berlin 1928, Springer)

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## MODULE SPECIFICATION

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This module has the following specification:

Name: prophilbert1  
Version: 1.00.00  
Rule version: 1.00.00  
Orgin: [http://www.meyling.com/principia/0\\_00\\_51/prophilbert1\\_1.00.00\\_1.00.00.qedeq](http://www.meyling.com/principia/0_00_51/prophilbert1_1.00.00_1.00.00.qedeq)

The following modules were used:

Name: propaxiom  
Version: 1.00.00  
Rule version: 1.00.00  
Orgin: [propaxiom\\_1.00.00\\_1.00.00.qedeq](#)  
pdf: [propaxiom\\_1.00.00\\_1.00.00.pdf](#)

Is used by the following modules:

Name: prophilbert2  
Version: 1.00.00  
Rule version: 1.00.00  
Orgin: [prophilbert2\\_1.00.00\\_1.00.00.qedeq](#)  
pdf: [prophilbert2\\_1.00.00\\_1.00.00.pdf](#)

First we prove a simple implication, that follows directly from the fourth axiom:

**Theorem 0.1.**

$$((P \Rightarrow Q) \Rightarrow ((A \Rightarrow P) \Rightarrow (A \Rightarrow Q)))$$

*Proof.*

1	$((P \Rightarrow Q) \Rightarrow ((A \vee P) \Rightarrow (A \vee Q)))$	add axiom axiom4
2	$((P \Rightarrow Q) \Rightarrow ((\neg A \vee P) \Rightarrow (\neg A \vee Q)))$	replace A by $\neg A$ in 1
3	$((P \Rightarrow Q) \Rightarrow ((A \Rightarrow P) \Rightarrow (\neg A \vee Q)))$	reverse abbreviation impl in 2 at occurrence 1
4	$((P \Rightarrow Q) \Rightarrow ((A \Rightarrow P) \Rightarrow (A \Rightarrow Q)))$	reverse abbreviation impl in 3 at occurrence 1

□

This proposition is the form for the Hypothetical Syllogism.

The self implication could be derived:

**Theorem 0.2.**

$$(P \Rightarrow P)$$

*Proof.*

1	$(P \Rightarrow (P \vee Q))$	add axiom axiom2
2	$(P \Rightarrow (P \vee P))$	replace Q by P in 1
3	$((P \vee P) \Rightarrow P)$	add axiom axiom1
4	$((P \Rightarrow Q) \Rightarrow ((A \Rightarrow P) \Rightarrow (A \Rightarrow Q)))$	add sentence hilb1
5	$((P \Rightarrow Q) \Rightarrow ((B \Rightarrow P) \Rightarrow (B \Rightarrow Q)))$	replace A by B in 4
6	$((P \Rightarrow C) \Rightarrow ((B \Rightarrow P) \Rightarrow (B \Rightarrow C)))$	replace Q by C in 5
7	$((D \Rightarrow C) \Rightarrow ((B \Rightarrow D) \Rightarrow (B \Rightarrow C)))$	replace P by D in 6
8	$((D \Rightarrow C) \Rightarrow ((P \Rightarrow D) \Rightarrow (P \Rightarrow C)))$	replace B by P in 7
9	$((D \Rightarrow P) \Rightarrow ((P \Rightarrow D) \Rightarrow (P \Rightarrow P)))$	replace C by P in 8
10	$((P \vee P) \Rightarrow P) \Rightarrow ((P \Rightarrow (P \vee P)) \Rightarrow (P \Rightarrow P))$	replace D by $(P \vee P)$ in 9
11	$((P \Rightarrow (P \vee P)) \Rightarrow (P \Rightarrow P))$	MP with 3, 10
12	$(P \Rightarrow P)$	MP with 2, 11

□

One form of the classical ‘tertium non datur’

**Theorem 0.3.**

$$(\neg P \vee P)$$

*Proof.*

1	$(P \Rightarrow P)$	add sentence hilb2
2	$(\neg P \vee P)$	use abbreviation impl in 1 at occurrence 1

□

The standard form of the excluded middle:

**Theorem 0.4.**

$$(P \vee \neg P)$$

*Proof.*

1	$(\neg P \vee P)$	add sentence hilb3
2	$((P \vee Q) \Rightarrow (Q \vee P))$	add axiom axiom3
3	$((P \vee A) \Rightarrow (A \vee P))$	replace $Q$ by $A$ in 2
4	$((B \vee A) \Rightarrow (A \vee B))$	replace $P$ by $B$ in 3
5	$((B \vee P) \Rightarrow (P \vee B))$	replace $A$ by $P$ in 4
6	$((\neg P \vee P) \Rightarrow (P \vee \neg P))$	replace $B$ by $\neg P$ in 5
7	$(P \vee \neg P)$	MP with 1, 6

□

Double negation is implicated:

**Theorem 0.5.**

$$(P \Rightarrow \neg\neg P)$$

*Proof.*

1	$(P \vee \neg P)$	add sentence hilb4
2	$(\neg P \vee \neg\neg P)$	replace $P$ by $\neg P$ in 1
3	$(P \Rightarrow \neg\neg P)$	reverse abbreviation impl in 2 at occurrence 1

□

The reverse is also true:

**Theorem 0.6.**

$$(\neg\neg P \Rightarrow P)$$

*Proof.*

1	$(P \Rightarrow \neg\neg P)$	add sentence hilb5
2	$(\neg P \Rightarrow \neg\neg\neg P)$	replace $P$ by $\neg P$ in 1
3	$((P \Rightarrow Q) \Rightarrow ((A \vee P) \Rightarrow (A \vee Q)))$	add axiom axiom4
4	$((P \Rightarrow Q) \Rightarrow ((B \vee P) \Rightarrow (B \vee Q)))$	replace $A$ by $B$ in 3
5	$((P \Rightarrow C) \Rightarrow ((B \vee P) \Rightarrow (B \vee C)))$	replace $Q$ by $C$ in 4
6	$((D \Rightarrow C) \Rightarrow ((B \vee D) \Rightarrow (B \vee C)))$	replace $P$ by $D$ in 5
7	$((D \Rightarrow C) \Rightarrow ((P \vee D) \Rightarrow (P \vee C)))$	replace $B$ by $P$ in 6
8	$((D \Rightarrow \neg\neg\neg P) \Rightarrow ((P \vee D) \Rightarrow (P \vee \neg\neg\neg P)))$	replace $C$ by $\neg\neg\neg P$ in 7
9	$((\neg P \Rightarrow \neg\neg\neg P) \Rightarrow ((P \vee \neg P) \Rightarrow (P \vee \neg\neg\neg P)))$	replace $D$ by $\neg P$ in 8
10	$((P \vee \neg P) \Rightarrow (P \vee \neg\neg\neg P))$	MP with 2, 9
11	$(P \vee \neg P)$	add sentence hilb4
12	$(P \vee \neg\neg\neg P)$	MP with 11, 10

13	$((P \vee Q) \Rightarrow (Q \vee P))$	add axiom axiom3
14	$((P \vee A) \Rightarrow (A \vee P))$	replace Q by A in 13
15	$((B \vee A) \Rightarrow (A \vee B))$	replace P by B in 14
16	$((B \vee \neg\neg P) \Rightarrow (\neg\neg P \vee B))$	replace A by $\neg\neg P$ in 15
17	$((P \vee \neg\neg P) \Rightarrow (\neg\neg P \vee P))$	replace B by P in 16
18	$(\neg\neg P \vee P)$	MP with 12, 17
19	$(\neg\neg P \Rightarrow P)$	reverse abbreviation impl in 18 at occurrence 1

□

The correct reverse of an implication:

**Theorem 0.7.**

$$((P \Rightarrow Q) \Rightarrow (\neg Q \Rightarrow \neg P))$$

*Proof.*

1	$(P \Rightarrow \neg\neg P)$	add sentence hilb5
2	$(Q \Rightarrow \neg\neg Q)$	replace P by Q in 1
3	$((P \Rightarrow Q) \Rightarrow ((A \vee P) \Rightarrow (A \vee Q)))$	add axiom axiom4
4	$((P \Rightarrow Q) \Rightarrow ((B \vee P) \Rightarrow (B \vee Q)))$	replace A by B in 3
5	$((P \Rightarrow C) \Rightarrow ((B \vee P) \Rightarrow (B \vee C)))$	replace Q by C in 4
6	$((D \Rightarrow C) \Rightarrow ((B \vee D) \Rightarrow (B \vee C)))$	replace P by D in 5
7	$((D \Rightarrow C) \Rightarrow ((\neg P \vee D) \Rightarrow (\neg P \vee C)))$	replace B by $\neg P$ in 6
8	$((D \Rightarrow \neg\neg Q) \Rightarrow ((\neg P \vee D) \Rightarrow (\neg P \vee \neg\neg Q)))$	replace C by $\neg\neg Q$ in 7
9	$((Q \Rightarrow \neg\neg Q) \Rightarrow ((\neg P \vee Q) \Rightarrow (\neg P \vee \neg\neg Q)))$	replace D by Q in 8
10	$(\neg\neg P \vee Q) \Rightarrow (\neg P \vee \neg\neg Q)$	MP with 2, 9
11	$((P \vee Q) \Rightarrow (Q \vee P))$	add axiom axiom3
12	$((P \vee A) \Rightarrow (A \vee P))$	replace Q by A in 11
13	$((B \vee A) \Rightarrow (A \vee B))$	replace P by B in 12
14	$((B \vee \neg\neg Q) \Rightarrow (\neg\neg Q \vee B))$	replace A by $\neg\neg Q$ in 13
15	$((\neg P \vee \neg\neg Q) \Rightarrow (\neg\neg Q \vee \neg P))$	replace B by $\neg P$ in 14
16	$((P \Rightarrow Q) \Rightarrow ((A \Rightarrow P) \Rightarrow (A \Rightarrow Q)))$	add sentence hilb1
17	$((P \Rightarrow Q) \Rightarrow ((B \Rightarrow P) \Rightarrow (B \Rightarrow Q)))$	replace A by B in 16
18	$((P \Rightarrow C) \Rightarrow ((B \Rightarrow P) \Rightarrow (B \Rightarrow C)))$	replace Q by C in 17
19	$((D \Rightarrow C) \Rightarrow ((B \Rightarrow D) \Rightarrow (B \Rightarrow C)))$	replace P by D in 18
20	$((D \Rightarrow C) \Rightarrow (((\neg P \vee Q) \Rightarrow D) \Rightarrow ((\neg P \vee Q) \Rightarrow C)))$	replace B by $(\neg P \vee Q)$ in 19
21	$((D \Rightarrow (\neg\neg Q \vee \neg P)) \Rightarrow (((\neg P \vee Q) \Rightarrow D) \Rightarrow ((\neg P \vee Q) \Rightarrow (\neg\neg Q \vee \neg P))))$	replace C by $(\neg\neg Q \vee \neg P)$ in 20
22	$((\neg\neg P \vee \neg\neg Q) \Rightarrow (\neg\neg Q \vee \neg P)) \Rightarrow (((\neg P \vee Q) \Rightarrow (\neg P \vee \neg\neg Q)) \Rightarrow ((\neg P \vee Q) \Rightarrow (\neg\neg Q \vee \neg P)))$	replace D by $(\neg P \vee \neg\neg Q)$ in 21
23	$((\neg\neg P \vee Q) \Rightarrow (\neg P \vee \neg\neg Q)) \Rightarrow ((\neg P \vee Q) \Rightarrow (\neg\neg Q \vee \neg P))$	MP with 15, 22
24	$((\neg P \vee Q) \Rightarrow (\neg\neg Q \vee \neg P))$	MP with 10, 23
25	$((P \Rightarrow Q) \Rightarrow (\neg\neg Q \vee \neg P))$	reverse abbreviation impl in 24 at occurrence 1
26	$((P \Rightarrow Q) \Rightarrow (\neg Q \Rightarrow \neg P))$	reverse abbreviation impl in 25 at occurrence 1

□

Definition of an Implication 1. part:

**Theorem 0.8.**

$$((P \Rightarrow Q) \Rightarrow (\neg P \vee Q))$$

*Proof.*

- 1  $(P \Rightarrow P)$
- 2  $(A \Rightarrow A)$
- 3  $((P \Rightarrow Q) \Rightarrow (P \Rightarrow Q))$
- 4  $((P \Rightarrow Q) \Rightarrow (\neg P \vee Q))$

add sentence **hilb2**  
 replace  $P$  by  $A$  in **1**  
 replace  $A$  by  $(P \Rightarrow Q)$  in **2**  
 use abbreviation **impl** in **3** at oc-  
 currence 3

□

Definition of an Implication 2. part:

**Theorem 0.9.**

$$((\neg P \vee Q) \Rightarrow (P \Rightarrow Q))$$

*Proof.*

- 1  $(P \Rightarrow P)$
- 2  $(A \Rightarrow A)$
- 3  $((P \Rightarrow Q) \Rightarrow (P \Rightarrow Q))$
- 4  $((\neg P \vee Q) \Rightarrow (P \Rightarrow Q))$

add sentence **hilb2**  
 replace  $P$  by  $A$  in **1**  
 replace  $A$  by  $(P \Rightarrow Q)$  in **2**  
 use abbreviation **impl** in **3** at oc-  
 currence 2

□

Definition of a Conjunction 1. part:

**Theorem 0.10.**

$$((P \wedge Q) \Rightarrow \neg(\neg P \vee \neg Q))$$

*Proof.*

- 1  $(P \Rightarrow P)$
- 2  $(A \Rightarrow A)$
- 3  $((P \wedge Q) \Rightarrow (P \wedge Q))$
- 4  $((P \wedge Q) \Rightarrow \neg(\neg P \vee \neg Q))$

add sentence **hilb2**  
 replace  $P$  by  $A$  in **1**  
 replace  $A$  by  $(P \wedge Q)$  in **2**  
 use abbreviation **and** in **3** at oc-  
 currence 2

□

Definition of a Conjunction 2. part:

**Theorem 0.11.**

$$(\neg(\neg P \vee \neg Q) \Rightarrow (P \wedge Q))$$

*Proof.*

1	$(P \Rightarrow P)$	add sentence hilb2
2	$(A \Rightarrow A)$	replace $P$ by $A$ in 1
3	$((P \wedge Q) \Rightarrow (P \wedge Q))$	replace $A$ by $(P \wedge Q)$ in 2
4	$(\neg(\neg P \vee \neg Q) \Rightarrow (P \wedge Q))$	use abbreviation and in 3 at occurrence 1

□

Definition of an Equivalence 1. part:

**Theorem 0.12.**

$$((P \Leftrightarrow Q) \Rightarrow ((P \Rightarrow Q) \wedge (Q \Rightarrow P)))$$

*Proof.*

1	$(P \Rightarrow P)$	add sentence hilb2
2	$(A \Rightarrow A)$	replace $P$ by $A$ in 1
3	$((P \Leftrightarrow Q) \Rightarrow (P \Leftrightarrow Q))$	replace $A$ by $(P \Leftrightarrow Q)$ in 2
4	$((P \Leftrightarrow Q) \Rightarrow ((P \Rightarrow Q) \wedge (Q \Rightarrow P)))$	use abbreviation equi in 3 at occurrence 2

□

Definition of an Equivalence 2. part:

**Theorem 0.13.**

$$(((P \Rightarrow Q) \wedge (Q \Rightarrow P)) \Rightarrow (P \Leftrightarrow Q))$$

*Proof.*

1	$(P \Rightarrow P)$	add sentence hilb2
2	$(A \Rightarrow A)$	replace $P$ by $A$ in 1
3	$((P \Leftrightarrow Q) \Rightarrow (P \Leftrightarrow Q))$	replace $A$ by $(P \Leftrightarrow Q)$ in 2
4	$((((P \Rightarrow Q) \wedge (Q \Rightarrow P)) \Rightarrow (P \Leftrightarrow Q)))$	use abbreviation equi in 3 at occurrence 1

□

A similar formulation for the second axiom:

**Theorem 0.14.**

$$(P \Rightarrow (Q \vee P))$$

*Proof.*

1	$(P \Rightarrow (P \vee Q))$	add axiom axiom2
2	$((P \vee Q) \Rightarrow (Q \vee P))$	add axiom axiom3
3	$((P \Rightarrow Q) \Rightarrow ((A \Rightarrow P) \Rightarrow (A \Rightarrow Q)))$	add sentence hilb1
4	$((P \Rightarrow Q) \Rightarrow ((B \Rightarrow P) \Rightarrow (B \Rightarrow Q)))$	replace $A$ by $B$ in 3
5	$((P \Rightarrow C) \Rightarrow ((B \Rightarrow P) \Rightarrow (B \Rightarrow C)))$	replace $Q$ by $C$ in 4
6	$((D \Rightarrow C) \Rightarrow ((B \Rightarrow D) \Rightarrow (B \Rightarrow C)))$	replace $P$ by $D$ in 5

7	$((D \Rightarrow C) \Rightarrow ((P \Rightarrow D) \Rightarrow (P \Rightarrow C)))$	replace $B$ by $P$ in 6
8	$((D \Rightarrow (Q \vee P)) \Rightarrow ((P \Rightarrow D) \Rightarrow (P \Rightarrow (Q \vee P))))$	replace $C$ by $(Q \vee P)$ in 7
9	$((P \vee Q) \Rightarrow (Q \vee P)) \Rightarrow ((P \Rightarrow (P \vee Q)) \Rightarrow (P \Rightarrow (Q \vee P)))$	replace $D$ by $(P \vee Q)$ in 8
10	$((P \Rightarrow (P \vee Q)) \Rightarrow (P \Rightarrow (Q \vee P)))$	MP with 2, 9
11	$(P \Rightarrow (Q \vee P))$	MP with 1, 10

□

A technical lemma (equal to the third axiom):

**Theorem 0.15.**

$$((P \vee Q) \Rightarrow (Q \vee P))$$

*Proof.*

1	$((P \vee Q) \Rightarrow (Q \vee P))$	add axiom axiom3
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□

And another technical lemma (similar to the third axiom):

**Theorem 0.16.**

$$((Q \vee P) \Rightarrow (P \vee Q))$$

*Proof.*

1	$((P \vee Q) \Rightarrow (Q \vee P))$	add axiom axiom3
2	$((P \vee A) \Rightarrow (A \vee P))$	replace $Q$ by $A$ in 1
3	$((B \vee A) \Rightarrow (A \vee B))$	replace $P$ by $B$ in 2
4	$((B \vee P) \Rightarrow (P \vee B))$	replace $A$ by $P$ in 3
5	$((Q \vee P) \Rightarrow (P \vee Q))$	replace $B$ by $Q$ in 4

□

A technical lemma (equal to the first axiom):

**Theorem 0.17.**

$$((P \vee P) \Rightarrow P)$$

*Proof.*

1	$((P \vee P) \Rightarrow P)$	add axiom axiom1
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□

A simple proposition that follows directly from the second axiom:

**Theorem 0.18.**

$$(P \Rightarrow (P \vee P))$$

*Proof.*

$$\begin{array}{l} 1 \quad (P \Rightarrow (P \vee Q)) \\ 2 \quad (P \Rightarrow (P \vee P)) \end{array}$$

add axiom axiom2  
replace Q by P in 1

□

This is a pre form for the associative law:

**Theorem 0.19.**

$$((P \vee (Q \vee A)) \Rightarrow (Q \vee (P \vee A)))$$

*Proof.*

$$\begin{array}{l} 1 \quad (P \Rightarrow (Q \vee P)) \\ 2 \quad (P \Rightarrow (B \vee P)) \\ 3 \quad (C \Rightarrow (B \vee C)) \\ 4 \quad (C \Rightarrow (P \vee C)) \\ 5 \quad (A \Rightarrow (P \vee A)) \\ 6 \quad ((P \Rightarrow Q) \Rightarrow ((A \vee P) \Rightarrow (A \vee Q))) \\ 7 \quad ((P \Rightarrow Q) \Rightarrow ((B \vee P) \Rightarrow (B \vee Q))) \\ 8 \quad ((P \Rightarrow C) \Rightarrow ((B \vee P) \Rightarrow (B \vee C))) \\ 9 \quad ((D \Rightarrow C) \Rightarrow ((B \vee D) \Rightarrow (B \vee C))) \\ 10 \quad ((D \Rightarrow C) \Rightarrow ((Q \vee D) \Rightarrow (Q \vee C))) \\ 11 \quad ((D \Rightarrow (P \vee A)) \Rightarrow ((Q \vee D) \Rightarrow (Q \vee (P \vee A)))) \\ 12 \quad ((A \Rightarrow (P \vee A)) \Rightarrow ((Q \vee A) \Rightarrow (Q \vee (P \vee A)))) \\ 13 \quad ((Q \vee A) \Rightarrow (Q \vee (P \vee A))) \\ 14 \quad ((D \Rightarrow C) \Rightarrow ((P \vee D) \Rightarrow (P \vee C))) \\ 15 \quad ((D \Rightarrow (Q \vee (P \vee A))) \Rightarrow ((P \vee D) \Rightarrow (P \vee (Q \vee (P \vee A)))))) \\ 16 \quad (((Q \vee A) \Rightarrow (Q \vee (P \vee A))) \Rightarrow ((P \vee (Q \vee A)) \Rightarrow (P \vee (Q \vee (P \vee A)))))) \\ 17 \quad ((P \vee (Q \vee A)) \Rightarrow (P \vee (Q \vee (P \vee A)))) \\ 18 \quad ((P \vee Q) \Rightarrow (Q \vee P)) \\ 19 \quad ((P \vee B) \Rightarrow (B \vee P)) \\ 20 \quad ((C \vee B) \Rightarrow (B \vee C)) \\ 21 \quad ((C \vee (Q \vee (P \vee A))) \Rightarrow ((Q \vee (P \vee A)) \vee C)) \\ \\ 22 \quad ((P \vee (Q \vee (P \vee A))) \Rightarrow ((Q \vee (P \vee A)) \vee P)) \\ 23 \quad ((P \Rightarrow Q) \Rightarrow (\neg P \vee Q)) \\ 24 \quad ((\neg P \vee Q) \Rightarrow (P \Rightarrow Q)) \\ 25 \quad ((D \Rightarrow C) \Rightarrow ((\neg(P \vee (Q \vee A)) \vee D) \Rightarrow (\neg(P \vee (Q \vee A)) \vee C))) \\ 26 \quad ((D \Rightarrow ((Q \vee (P \vee A)) \vee P)) \Rightarrow ((\neg(P \vee (Q \vee A)) \vee D) \Rightarrow (\neg(P \vee (Q \vee A)) \vee ((Q \vee (P \vee A)) \vee P)))) \\ 27 \quad (((P \vee (Q \vee (P \vee A))) \Rightarrow ((Q \vee (P \vee A)) \vee P)) \Rightarrow ((\neg(P \vee (Q \vee A)) \vee (P \vee (Q \vee (P \vee A)))) \Rightarrow (\neg(P \vee (Q \vee A)) \vee ((Q \vee (P \vee A)) \vee P)))) \\ 28 \quad ((\neg(P \vee (Q \vee A)) \vee (P \vee (Q \vee (P \vee A)))) \Rightarrow (\neg(P \vee (Q \vee A)) \vee ((Q \vee (P \vee A)) \vee P))) \\ 29 \quad ((P \Rightarrow B) \Rightarrow (\neg P \vee B)) \end{array}$$

add sentence hilb8  
replace Q by B in 1  
replace P by C in 2  
replace B by P in 3  
replace C by A in 4  
add axiom axiom4  
replace A by B in 6  
replace Q by C in 7  
replace P by D in 8  
replace B by Q in 9  
replace C by (P ∨ A) in 10  
replace D by A in 11  
MP with 5, 12  
replace B by P in 9  
replace C by (Q ∨ (P ∨ A)) in 14  
  
replace D by (Q ∨ A) in 15  
  
MP with 13, 16  
add sentence hilb9  
replace Q by B in 18  
replace P by C in 19  
replace B by (Q ∨ (P ∨ A)) in 20  
  
replace C by P in 21  
add sentence defimpl1  
add sentence defimpl2  
replace B by ¬(P ∨ (Q ∨ A)) in 9  
  
replace C by ((Q ∨ (P ∨ A)) ∨ P) in 25  
replace D by (P ∨ (Q ∨ (P ∨ A))) in 26  
  
MP with 22, 27  
  
replace Q by B in 23

30	$((C \Rightarrow B) \Rightarrow (\neg C \vee B))$	replace $P$ by $C$ in 29
31	$((C \Rightarrow (P \vee (Q \vee (P \vee A)))) \Rightarrow (\neg C \vee (P \vee (Q \vee (P \vee A))))$	replace $B$ by $(P \vee (Q \vee (P \vee A)))$ in 30
32	$(( (P \vee (Q \vee A)) \Rightarrow (P \vee (Q \vee (P \vee A))) \Rightarrow (\neg(P \vee (Q \vee A)) \vee (P \vee (Q \vee (P \vee A))))$	replace $C$ by $(P \vee (Q \vee A))$ in 31
33	$((P \Rightarrow Q) \Rightarrow ((A \Rightarrow P) \Rightarrow (A \Rightarrow Q)))$	add sentence hilb1
34	$((P \Rightarrow Q) \Rightarrow ((B \Rightarrow P) \Rightarrow (B \Rightarrow Q)))$	replace $A$ by $B$ in 33
35	$((P \Rightarrow C) \Rightarrow ((B \Rightarrow P) \Rightarrow (B \Rightarrow C)))$	replace $Q$ by $C$ in 34
36	$((D \Rightarrow C) \Rightarrow ((B \Rightarrow D) \Rightarrow (B \Rightarrow C)))$	replace $P$ by $D$ in 35
37	$((D \Rightarrow C) \Rightarrow (((P \vee (Q \vee A)) \Rightarrow (P \vee (Q \vee (P \vee A)))) \Rightarrow D) \Rightarrow (((P \vee (Q \vee A)) \Rightarrow (P \vee (Q \vee (P \vee A)))) \Rightarrow C))$	replace $B$ by $((P \vee (Q \vee A)) \Rightarrow (P \vee (Q \vee (P \vee A))))$ in 36
38	$((D \Rightarrow (\neg(P \vee (Q \vee A)) \vee ((Q \vee (P \vee A)) \vee P)) \Rightarrow (((P \vee (Q \vee A)) \Rightarrow (P \vee (Q \vee (P \vee A)))) \Rightarrow D) \Rightarrow (((P \vee (Q \vee A)) \Rightarrow (P \vee (Q \vee (P \vee A)))) \Rightarrow (\neg(P \vee (Q \vee A)) \vee ((Q \vee (P \vee A)) \vee P))))$	replace $C$ by $(\neg(P \vee (Q \vee A)) \vee ((Q \vee (P \vee A)) \vee P))$ in 37
39	$(( (\neg(P \vee (Q \vee A)) \vee (P \vee (Q \vee (P \vee A)))) \Rightarrow (\neg(P \vee (Q \vee A)) \vee ((Q \vee (P \vee A)) \vee P)) \Rightarrow (((P \vee (Q \vee A)) \Rightarrow (P \vee (Q \vee (P \vee A)))) \Rightarrow (\neg(P \vee (Q \vee A)) \vee (P \vee (Q \vee (P \vee A)))) \Rightarrow (((P \vee (Q \vee A)) \Rightarrow (P \vee (Q \vee (P \vee A)))) \Rightarrow (\neg(P \vee (Q \vee A)) \vee ((Q \vee (P \vee A)) \vee P))))$	replace $D$ by $(\neg(P \vee (Q \vee A)) \vee (P \vee (Q \vee (P \vee A))))$ in 38
40	$(( ((P \vee (Q \vee A)) \Rightarrow (P \vee (Q \vee (P \vee A))) \Rightarrow (\neg(P \vee (Q \vee A)) \vee (P \vee (Q \vee (P \vee A)))) \Rightarrow (((P \vee (Q \vee A)) \Rightarrow (P \vee (Q \vee (P \vee A)))) \Rightarrow (\neg(P \vee (Q \vee A)) \vee ((Q \vee (P \vee A)) \vee P))))$	MP with 28, 39
41	$(( (P \vee (Q \vee A)) \Rightarrow (P \vee (Q \vee (P \vee A))) \Rightarrow (\neg(P \vee (Q \vee A)) \vee ((Q \vee (P \vee A)) \vee P))$	MP with 32, 40
42	$((\neg P \vee B) \Rightarrow (P \Rightarrow B))$	replace $Q$ by $B$ in 24
43	$((\neg C \vee B) \Rightarrow (C \Rightarrow B))$	replace $P$ by $C$ in 42
44	$((\neg C \vee ((Q \vee (P \vee A)) \vee P)) \Rightarrow (C \Rightarrow ((Q \vee (P \vee A)) \vee P)))$	replace $B$ by $((Q \vee (P \vee A)) \vee P)$ in 43
45	$(( (\neg(P \vee (Q \vee A)) \vee ((Q \vee (P \vee A)) \vee P)) \Rightarrow ((P \vee (Q \vee A)) \Rightarrow ((Q \vee (P \vee A)) \vee P)))$	replace $C$ by $(P \vee (Q \vee A))$ in 44
46	$((D \Rightarrow ((P \vee (Q \vee A)) \Rightarrow ((Q \vee (P \vee A)) \vee P)) \Rightarrow (((P \vee (Q \vee A)) \Rightarrow (P \vee (Q \vee (P \vee A)))) \Rightarrow D) \Rightarrow (((P \vee (Q \vee A)) \Rightarrow (P \vee (Q \vee (P \vee A)))) \Rightarrow ((P \vee (Q \vee A)) \Rightarrow ((Q \vee (P \vee A)) \vee P))))$	replace $C$ by $((P \vee (Q \vee A)) \Rightarrow ((Q \vee (P \vee A)) \vee P))$ in 37
47	$(( (\neg(P \vee (Q \vee A)) \vee ((Q \vee (P \vee A)) \vee P)) \Rightarrow ((P \vee (Q \vee A)) \Rightarrow ((Q \vee (P \vee A)) \vee P)) \Rightarrow (((P \vee (Q \vee A)) \Rightarrow (P \vee (Q \vee (P \vee A)))) \Rightarrow (\neg(P \vee (Q \vee A)) \vee ((Q \vee (P \vee A)) \vee P)) \Rightarrow (((P \vee (Q \vee A)) \Rightarrow (P \vee (Q \vee (P \vee A)))) \Rightarrow ((P \vee (Q \vee A)) \Rightarrow ((Q \vee (P \vee A)) \vee P))))$	replace $D$ by $(\neg(P \vee (Q \vee A)) \vee ((Q \vee (P \vee A)) \vee P))$ in 46
48	$(( ((P \vee (Q \vee A)) \Rightarrow (P \vee (Q \vee (P \vee A))) \Rightarrow (\neg(P \vee (Q \vee A)) \vee ((Q \vee (P \vee A)) \vee P)) \Rightarrow (((P \vee (Q \vee A)) \Rightarrow (P \vee (Q \vee (P \vee A)))) \Rightarrow ((P \vee (Q \vee A)) \Rightarrow ((Q \vee (P \vee A)) \vee P))))$	MP with 45, 47
49	$(( (P \vee (Q \vee A)) \Rightarrow (P \vee (Q \vee (P \vee A))) \Rightarrow ((P \vee (Q \vee A)) \Rightarrow ((Q \vee (P \vee A)) \vee P)))$	MP with 41, 48
50	$((P \vee (Q \vee A)) \Rightarrow ((Q \vee (P \vee A)) \vee P))$	MP with 17, 49
51	$((P \vee A) \Rightarrow (Q \vee (P \vee A)))$	replace $P$ by $(P \vee A)$ in 1

52	$(P \Rightarrow (P \vee Q))$	add axiom axiom2
53	$(P \Rightarrow (P \vee A))$	replace $Q$ by $A$ in 52
54	$((D \Rightarrow C) \Rightarrow ((P \Rightarrow D) \Rightarrow (P \Rightarrow C)))$	replace $B$ by $P$ in 36
55	$((D \Rightarrow (Q \vee (P \vee A))) \Rightarrow ((P \Rightarrow D) \Rightarrow (P \Rightarrow (Q \vee (P \vee A)))))$	replace $C$ by $(Q \vee (P \vee A))$ in 54
56	$((P \vee A) \Rightarrow (Q \vee (P \vee A))) \Rightarrow ((P \Rightarrow (P \vee A)) \Rightarrow (P \Rightarrow (Q \vee (P \vee A))))$	replace $D$ by $(P \vee A)$ in 55
57	$((P \Rightarrow (P \vee A)) \Rightarrow (P \Rightarrow (Q \vee (P \vee A))))$	MP with 51, 56
58	$(P \Rightarrow (Q \vee (P \vee A)))$	MP with 53, 57
59	$((D \Rightarrow C) \Rightarrow (((Q \vee (P \vee A)) \vee D) \Rightarrow ((Q \vee (P \vee A)) \vee C)))$	replace $B$ by $(Q \vee (P \vee A))$ in 9
60	$((D \Rightarrow (Q \vee (P \vee A))) \Rightarrow (((Q \vee (P \vee A)) \vee D) \Rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))))$	replace $C$ by $(Q \vee (P \vee A))$ in 59
61	$((P \Rightarrow (Q \vee (P \vee A))) \Rightarrow (((Q \vee (P \vee A)) \vee P) \Rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))))$	replace $D$ by $P$ in 60
62	$((Q \vee (P \vee A)) \vee P) \Rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))$	MP with 58, 61
63	$(P \vee P) \Rightarrow P$	add sentence hilb11
64	$(B \vee B) \Rightarrow B$	replace $P$ by $B$ in 63
65	$((Q \vee (P \vee A)) \vee (Q \vee (P \vee A))) \Rightarrow (Q \vee (P \vee A))$	replace $B$ by $(Q \vee (P \vee A))$ in 64
66	$((D \Rightarrow C) \Rightarrow ((\neg((Q \vee (P \vee A)) \vee P) \vee D) \Rightarrow (\neg((Q \vee (P \vee A)) \vee P) \vee C)))$	replace $B$ by $\neg((Q \vee (P \vee A)) \vee P)$ in 9
67	$((D \Rightarrow (Q \vee (P \vee A))) \Rightarrow ((\neg((Q \vee (P \vee A)) \vee P) \vee D) \Rightarrow (\neg((Q \vee (P \vee A)) \vee P) \vee (Q \vee (P \vee A)))))$	replace $C$ by $(Q \vee (P \vee A))$ in 66
68	$((Q \vee (P \vee A)) \vee (Q \vee (P \vee A))) \Rightarrow (Q \vee (P \vee A)) \Rightarrow ((\neg((Q \vee (P \vee A)) \vee P) \vee ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))) \Rightarrow (\neg((Q \vee (P \vee A)) \vee P) \vee (Q \vee (P \vee A))))$	replace $D$ by $((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))$ in 67
69	$((\neg((Q \vee (P \vee A)) \vee P) \vee ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))) \Rightarrow (\neg((Q \vee (P \vee A)) \vee P) \vee (Q \vee (P \vee A))))$	MP with 65, 68
70	$((C \Rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))) \Rightarrow (\neg C \vee ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A))))$	replace $B$ by $((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))$ in 30
71	$((Q \vee (P \vee A)) \vee P) \Rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A))) \Rightarrow (\neg((Q \vee (P \vee A)) \vee P) \vee ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A))))$	replace $C$ by $((Q \vee (P \vee A)) \vee P)$ in 70
72	$((D \Rightarrow C) \Rightarrow (((Q \vee (P \vee A)) \vee P) \Rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))) \Rightarrow D) \Rightarrow (((Q \vee (P \vee A)) \vee P) \Rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))) \Rightarrow C)$	replace $B$ by $((Q \vee (P \vee A)) \vee P) \Rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))$ in 36
73	$((D \Rightarrow (\neg((Q \vee (P \vee A)) \vee P) \vee (Q \vee (P \vee A)))) \Rightarrow (((Q \vee (P \vee A)) \vee P) \Rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))) \Rightarrow D) \Rightarrow (((Q \vee (P \vee A)) \vee P) \Rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))) \Rightarrow (\neg((Q \vee (P \vee A)) \vee P) \vee (Q \vee (P \vee A))))$	replace $C$ by $(\neg((Q \vee (P \vee A)) \vee P) \vee (Q \vee (P \vee A)))$ in 72
74	$((\neg((Q \vee (P \vee A)) \vee P) \vee ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))) \Rightarrow (\neg((Q \vee (P \vee A)) \vee P) \vee (Q \vee (P \vee A)))) \Rightarrow (((Q \vee (P \vee A)) \vee P) \Rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))) \Rightarrow (\neg((Q \vee (P \vee A)) \vee P) \vee (Q \vee (P \vee A)))) \Rightarrow (((Q \vee (P \vee A)) \vee P) \Rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))) \Rightarrow (\neg((Q \vee (P \vee A)) \vee P) \vee (Q \vee (P \vee A))))$	replace $D$ by $(\neg((Q \vee (P \vee A)) \vee P) \vee ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A))))$ in 73

75	$\begin{aligned} & (((((Q \vee (P \vee A)) \vee P) \Rightarrow ((Q \vee (P \vee A)) \vee (Q \vee \\ & (P \vee A)))) \Rightarrow (\neg((Q \vee (P \vee A)) \vee P) \vee ((Q \vee (P \vee \\ & A)) \vee (Q \vee (P \vee A)))) \Rightarrow (((Q \vee (P \vee A)) \vee P) \Rightarrow \\ & ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))) \Rightarrow (\neg((Q \vee (P \vee \\ & A)) \vee P) \vee (Q \vee (P \vee A)))) \end{aligned}$	MP with 69, 74
76	$\begin{aligned} & (((((Q \vee (P \vee A)) \vee P) \Rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee \\ & A)))) \Rightarrow (\neg((Q \vee (P \vee A)) \vee P) \vee (Q \vee (P \vee A)))) \end{aligned}$	MP with 71, 75
77	$((\neg C \vee (Q \vee (P \vee A))) \Rightarrow (C \Rightarrow (Q \vee (P \vee A))))$	replace $B$ by $(Q \vee (P \vee A))$ in 43
78	$\begin{aligned} & ((\neg((Q \vee (P \vee A)) \vee P) \vee (Q \vee (P \vee A))) \Rightarrow \\ & (((Q \vee (P \vee A)) \vee P) \Rightarrow (Q \vee (P \vee A)))) \end{aligned}$	replace $C$ by $((Q \vee (P \vee A)) \vee P)$ in 77
79	$\begin{aligned} & ((D \Rightarrow (((Q \vee (P \vee A)) \vee P) \Rightarrow (Q \vee (P \vee A)))) \Rightarrow \\ & (((((Q \vee (P \vee A)) \vee P) \Rightarrow ((Q \vee (P \vee A)) \vee (Q \vee \\ & (P \vee A)))) \Rightarrow D) \Rightarrow (((Q \vee (P \vee A)) \vee P) \Rightarrow \\ & ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))) \Rightarrow (((Q \vee (P \vee \\ & A)) \vee P) \Rightarrow (Q \vee (P \vee A)))) \end{aligned}$	replace $C$ by $((Q \vee (P \vee A)) \vee P) \Rightarrow (Q \vee (P \vee A))$ in 72
80	$\begin{aligned} & (((\neg((Q \vee (P \vee A)) \vee P) \vee (Q \vee (P \vee A))) \Rightarrow \\ & (((Q \vee (P \vee A)) \vee P) \Rightarrow (Q \vee (P \vee A)))) \Rightarrow (((((Q \vee \\ & (P \vee A)) \vee P) \Rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))) \Rightarrow \\ & (\neg((Q \vee (P \vee A)) \vee P) \vee (Q \vee (P \vee A)))) \Rightarrow (((Q \vee \\ & (P \vee A)) \vee P) \Rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))) \Rightarrow \\ & (((Q \vee (P \vee A)) \vee P) \Rightarrow (Q \vee (P \vee A)))) \end{aligned}$	replace $D$ by $(\neg((Q \vee (P \vee A)) \vee P) \vee (Q \vee (P \vee A)))$ in 79
81	$\begin{aligned} & (((((Q \vee (P \vee A)) \vee P) \Rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee \\ & A)))) \Rightarrow (\neg((Q \vee (P \vee A)) \vee P) \vee (Q \vee (P \vee A)))) \Rightarrow \\ & (((((Q \vee (P \vee A)) \vee P) \Rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee \\ & A)))) \Rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))) \Rightarrow \\ & (((Q \vee (P \vee A)) \vee P) \Rightarrow (Q \vee (P \vee A)))) \end{aligned}$	MP with 78, 80
82	$\begin{aligned} & (((((Q \vee (P \vee A)) \vee P) \Rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee \\ & A)))) \Rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))) \Rightarrow \\ & (((Q \vee (P \vee A)) \vee P) \Rightarrow (Q \vee (P \vee A)))) \end{aligned}$	MP with 76, 81
83	$(((Q \vee (P \vee A)) \vee P) \Rightarrow (Q \vee (P \vee A)))$	MP with 62, 82
84	$((D \Rightarrow C) \Rightarrow (((P \vee (Q \vee A)) \Rightarrow D) \Rightarrow ((P \vee (Q \vee A)) \Rightarrow C)))$	replace $B$ by $(P \vee (Q \vee A))$ in 36
85	$((D \Rightarrow (Q \vee (P \vee A))) \Rightarrow (((P \vee (Q \vee A)) \Rightarrow D) \Rightarrow ((P \vee (Q \vee A)) \Rightarrow (Q \vee (P \vee A))))$	replace $C$ by $(Q \vee (P \vee A))$ in 84
86	$\begin{aligned} & (((((Q \vee (P \vee A)) \vee P) \Rightarrow (Q \vee (P \vee A))) \Rightarrow \\ & (((P \vee (Q \vee A)) \Rightarrow ((Q \vee (P \vee A)) \vee P)) \Rightarrow \\ & ((P \vee (Q \vee A)) \Rightarrow (Q \vee (P \vee A)))) \end{aligned}$	replace $D$ by $((Q \vee (P \vee A)) \vee P)$ in 85
87	$\begin{aligned} & (((P \vee (Q \vee A)) \Rightarrow ((Q \vee (P \vee A)) \vee P)) \Rightarrow \\ & ((P \vee (Q \vee A)) \Rightarrow (Q \vee (P \vee A)))) \end{aligned}$	MP with 83, 86
88	$((P \vee (Q \vee A)) \Rightarrow (Q \vee (P \vee A)))$	MP with 50, 87

□

The associative law for the disjunction (first direction):

**Theorem 0.20.**

$$((P \vee (Q \vee A)) \Rightarrow ((P \vee Q) \vee A))$$

*Proof.*

1	$((P \vee Q) \Rightarrow (Q \vee P))$	add sentence hilb9
2	$((P \vee B) \Rightarrow (B \vee P))$	replace $Q$ by $B$ in 1
3	$((C \vee B) \Rightarrow (B \vee C))$	replace $P$ by $C$ in 2

4	$((C \vee A) \Rightarrow (A \vee C))$	replace $B$ by $A$ in 3
5	$((Q \vee A) \Rightarrow (A \vee Q))$	replace $C$ by $Q$ in 4
6	$((P \Rightarrow Q) \Rightarrow ((A \vee P) \Rightarrow (A \vee Q)))$	add axiom axiom4
7	$((P \Rightarrow Q) \Rightarrow ((B \vee P) \Rightarrow (B \vee Q)))$	replace $A$ by $B$ in 6
8	$((P \Rightarrow C) \Rightarrow ((B \vee P) \Rightarrow (B \vee C)))$	replace $Q$ by $C$ in 7
9	$((D \Rightarrow C) \Rightarrow ((B \vee D) \Rightarrow (B \vee C)))$	replace $P$ by $D$ in 8
10	$((D \Rightarrow C) \Rightarrow ((P \vee D) \Rightarrow (P \vee C)))$	replace $B$ by $P$ in 9
11	$((D \Rightarrow (A \vee Q)) \Rightarrow ((P \vee D) \Rightarrow (P \vee (A \vee Q))))$	replace $C$ by $(A \vee Q)$ in 10
12	$((((Q \vee A) \Rightarrow (A \vee Q)) \Rightarrow ((P \vee (Q \vee A)) \Rightarrow (P \vee (A \vee Q))))$	replace $D$ by $(Q \vee A)$ in 11
13	$((P \vee (Q \vee A)) \Rightarrow (P \vee (A \vee Q)))$	MP with 5, 12
14	$((P \Rightarrow Q) \Rightarrow (\neg P \vee Q))$	add sentence defimpl1
15	$((\neg P \vee Q) \Rightarrow (P \Rightarrow Q))$	add sentence defimpl2
16	$((P \Rightarrow B) \Rightarrow (\neg P \vee B))$	replace $Q$ by $B$ in 14
17	$((C \Rightarrow B) \Rightarrow (\neg C \vee B))$	replace $P$ by $C$ in 16
18	$((P \Rightarrow Q) \Rightarrow ((A \Rightarrow P) \Rightarrow (A \Rightarrow Q)))$	add sentence hilb1
19	$((P \Rightarrow Q) \Rightarrow ((B \Rightarrow P) \Rightarrow (B \Rightarrow Q)))$	replace $A$ by $B$ in 18
20	$((P \Rightarrow C) \Rightarrow ((B \Rightarrow P) \Rightarrow (B \Rightarrow C)))$	replace $Q$ by $C$ in 19
21	$((D \Rightarrow C) \Rightarrow ((B \Rightarrow D) \Rightarrow (B \Rightarrow C)))$	replace $P$ by $D$ in 20
22	$((\neg P \vee B) \Rightarrow (P \Rightarrow B))$	replace $Q$ by $B$ in 15
23	$((\neg C \vee B) \Rightarrow (C \Rightarrow B))$	replace $P$ by $C$ in 22
24	$((P \vee (Q \vee A)) \Rightarrow (Q \vee (P \vee A)))$	add sentence hilb13
25	$((P \vee (Q \vee B)) \Rightarrow (Q \vee (P \vee B)))$	replace $A$ by $B$ in 24
26	$((P \vee (C \vee B)) \Rightarrow (C \vee (P \vee B)))$	replace $Q$ by $C$ in 25
27	$((D \vee (C \vee B)) \Rightarrow (C \vee (D \vee B)))$	replace $P$ by $D$ in 26
28	$((D \vee (C \vee Q)) \Rightarrow (C \vee (D \vee Q)))$	replace $B$ by $Q$ in 27
29	$((D \vee (A \vee Q)) \Rightarrow (A \vee (D \vee Q)))$	replace $C$ by $A$ in 28
30	$((P \vee (A \vee Q)) \Rightarrow (A \vee (P \vee Q)))$	replace $D$ by $P$ in 29
31	$((D \Rightarrow C) \Rightarrow (((P \vee (Q \vee A)) \Rightarrow D) \Rightarrow ((P \vee (Q \vee A)) \Rightarrow C)))$	replace $B$ by $(P \vee (Q \vee A))$ in 21
32	$((D \Rightarrow (A \vee (P \vee Q))) \Rightarrow (((P \vee (Q \vee A)) \Rightarrow D) \Rightarrow ((P \vee (Q \vee A)) \Rightarrow (A \vee (P \vee Q))))$	replace $C$ by $(A \vee (P \vee Q))$ in 31
33	$((((P \vee (A \vee Q)) \Rightarrow (A \vee (P \vee Q))) \Rightarrow (((P \vee (Q \vee A)) \Rightarrow (P \vee (A \vee Q))) \Rightarrow ((P \vee (Q \vee A)) \Rightarrow (A \vee (P \vee Q))))$	replace $D$ by $(P \vee (A \vee Q))$ in 32
34	$((((P \vee (Q \vee A)) \Rightarrow (P \vee (A \vee Q))) \Rightarrow ((P \vee (Q \vee A)) \Rightarrow (A \vee (P \vee Q))))$	MP with 30, 33
35	$((P \vee (Q \vee A)) \Rightarrow (A \vee (P \vee Q)))$	MP with 13, 34
36	$((C \vee (P \vee Q)) \Rightarrow ((P \vee Q) \vee C))$	replace $B$ by $(P \vee Q)$ in 3
37	$((A \vee (P \vee Q)) \Rightarrow ((P \vee Q) \vee A))$	replace $C$ by $A$ in 36
38	$((D \Rightarrow C) \Rightarrow ((\neg(P \vee (Q \vee A)) \vee D) \Rightarrow (\neg(P \vee (Q \vee A)) \vee C)))$	replace $B$ by $\neg(P \vee (Q \vee A))$ in 9
39	$((D \Rightarrow ((P \vee Q) \vee A)) \Rightarrow ((\neg(P \vee (Q \vee A)) \vee D) \Rightarrow (\neg(P \vee (Q \vee A)) \vee ((P \vee Q) \vee A))))$	replace $C$ by $((P \vee Q) \vee A)$ in 38
40	$((A \vee (P \vee Q)) \Rightarrow ((P \vee Q) \vee A)) \Rightarrow ((\neg(P \vee (Q \vee A)) \vee (A \vee (P \vee Q))) \Rightarrow (\neg(P \vee (Q \vee A)) \vee ((P \vee Q) \vee A)))$	replace $D$ by $(A \vee (P \vee Q))$ in 39
41	$((\neg(P \vee (Q \vee A)) \vee (A \vee (P \vee Q))) \Rightarrow (\neg(P \vee (Q \vee A)) \vee ((P \vee Q) \vee A)))$	MP with 37, 40
42	$((C \Rightarrow (A \vee (P \vee Q))) \Rightarrow (\neg C \vee (A \vee (P \vee Q))))$	replace $B$ by $(A \vee (P \vee Q))$ in 17

43	$((P \vee (Q \vee A)) \Rightarrow (A \vee (P \vee Q))) \Rightarrow \neg(P \vee (Q \vee A)) \vee (A \vee (P \vee Q))$	replace $C$ by $(P \vee (Q \vee A))$ in 42
44	$((D \Rightarrow C) \Rightarrow (((P \vee (Q \vee A)) \Rightarrow (A \vee (P \vee Q))) \Rightarrow D) \Rightarrow (((P \vee (Q \vee A)) \Rightarrow (A \vee (P \vee Q))) \Rightarrow C))$	replace $B$ by $((P \vee (Q \vee A)) \Rightarrow (A \vee (P \vee Q)))$ in 21
45	$((D \Rightarrow \neg(P \vee (Q \vee A)) \vee ((P \vee Q) \vee A)) \Rightarrow (((P \vee (Q \vee A)) \Rightarrow (A \vee (P \vee Q))) \Rightarrow D) \Rightarrow (((P \vee (Q \vee A)) \Rightarrow (A \vee (P \vee Q))) \Rightarrow \neg(P \vee (Q \vee A)) \vee ((P \vee Q) \vee A)))$	replace $C$ by $\neg(P \vee (Q \vee A)) \vee ((P \vee Q) \vee A)$ in 44
46	$((\neg(P \vee (Q \vee A)) \vee (A \vee (P \vee Q))) \Rightarrow \neg(P \vee (Q \vee A)) \vee ((P \vee Q) \vee A)) \Rightarrow (((P \vee (Q \vee A)) \Rightarrow (A \vee (P \vee Q))) \Rightarrow \neg(P \vee (Q \vee A)) \vee (A \vee (P \vee Q))) \Rightarrow ((P \vee (Q \vee A)) \Rightarrow (A \vee (P \vee Q))) \Rightarrow \neg(P \vee (Q \vee A)) \vee ((P \vee Q) \vee A))$	replace $D$ by $\neg(P \vee (Q \vee A)) \vee (A \vee (P \vee Q))$ in 45
47	$((P \vee (Q \vee A)) \Rightarrow (A \vee (P \vee Q))) \Rightarrow \neg(P \vee (Q \vee A)) \vee (A \vee (P \vee Q))$	MP with 41, 46
48	$((P \vee (Q \vee A)) \Rightarrow (A \vee (P \vee Q))) \Rightarrow \neg(P \vee (Q \vee A)) \vee ((P \vee Q) \vee A)$	MP with 43, 47
49	$(\neg C \vee ((P \vee Q) \vee A)) \Rightarrow (C \Rightarrow ((P \vee Q) \vee A))$	replace $B$ by $((P \vee Q) \vee A)$ in 23
50	$(\neg(P \vee (Q \vee A)) \vee ((P \vee Q) \vee A)) \Rightarrow ((P \vee (Q \vee A)) \Rightarrow ((P \vee Q) \vee A))$	replace $C$ by $(P \vee (Q \vee A))$ in 49
51	$((D \Rightarrow ((P \vee (Q \vee A)) \Rightarrow ((P \vee Q) \vee A))) \Rightarrow (((P \vee (Q \vee A)) \Rightarrow (A \vee (P \vee Q))) \Rightarrow D) \Rightarrow (((P \vee (Q \vee A)) \Rightarrow (A \vee (P \vee Q))) \Rightarrow ((P \vee (Q \vee A)) \Rightarrow ((P \vee Q) \vee A)))$	replace $C$ by $((P \vee (Q \vee A)) \Rightarrow ((P \vee Q) \vee A))$ in 44
52	$((\neg(P \vee (Q \vee A)) \vee ((P \vee Q) \vee A)) \Rightarrow ((P \vee (Q \vee A)) \Rightarrow ((P \vee Q) \vee A))) \Rightarrow (((P \vee (Q \vee A)) \Rightarrow (A \vee (P \vee Q))) \Rightarrow \neg(P \vee (Q \vee A)) \vee ((P \vee Q) \vee A)) \Rightarrow (((P \vee (Q \vee A)) \Rightarrow (A \vee (P \vee Q))) \Rightarrow ((P \vee (Q \vee A)) \Rightarrow ((P \vee Q) \vee A)))$	replace $D$ by $\neg(P \vee (Q \vee A)) \vee ((P \vee Q) \vee A)$ in 51
53	$((P \vee (Q \vee A)) \Rightarrow (A \vee (P \vee Q))) \Rightarrow \neg(P \vee (Q \vee A)) \vee ((P \vee Q) \vee A)$	MP with 50, 52
54	$((P \vee (Q \vee A)) \Rightarrow (A \vee (P \vee Q))) \Rightarrow ((P \vee (Q \vee A)) \Rightarrow ((P \vee Q) \vee A))$	MP with 48, 53
55	$(P \vee (Q \vee A)) \Rightarrow ((P \vee Q) \vee A)$	MP with 35, 54

□

The associative law for the disjunction (second direction):

**Theorem 0.21.**

$$(((P \vee Q) \vee A) \Rightarrow (P \vee (Q \vee A)))$$

*Proof.*

1	$((P \vee (Q \vee A)) \Rightarrow ((P \vee Q) \vee A))$	add sentence hilb14
2	$((P \vee (Q \vee B)) \Rightarrow ((P \vee Q) \vee B))$	replace $A$ by $B$ in 1
3	$((P \vee (C \vee B)) \Rightarrow ((P \vee C) \vee B))$	replace $Q$ by $C$ in 2
4	$((D \vee (C \vee B)) \Rightarrow ((D \vee C) \vee B))$	replace $P$ by $D$ in 3
5	$((D \vee (C \vee P)) \Rightarrow ((D \vee C) \vee P))$	replace $B$ by $P$ in 4

6	$((D \vee (Q \vee P)) \Rightarrow ((D \vee Q) \vee P))$	replace $C$ by $Q$ in 5
7	$((A \vee (Q \vee P)) \Rightarrow ((A \vee Q) \vee P))$	replace $D$ by $A$ in 6
8	$((Q \vee P) \Rightarrow (P \vee Q))$	add sentence <b>hilb10</b>
9	$((B \vee P) \Rightarrow (P \vee B))$	replace $Q$ by $B$ in 8
10	$((B \vee C) \Rightarrow (C \vee B))$	replace $P$ by $C$ in 9
11	$((((Q \vee P) \vee C) \Rightarrow (C \vee (Q \vee P)))$	replace $B$ by $(Q \vee P)$ in 10
12	$((((Q \vee P) \vee A) \Rightarrow (A \vee (Q \vee P)))$	replace $C$ by $A$ in 11
13	$((P \Rightarrow Q) \Rightarrow (\neg P \vee Q))$	add sentence <b>defimpl1</b>
14	$((\neg P \vee Q) \Rightarrow (P \Rightarrow Q))$	add sentence <b>defimpl2</b>
15	$((P \Rightarrow Q) \Rightarrow (\neg Q \Rightarrow \neg P))$	add sentence <b>hilb7</b>
16	$((P \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg P))$	replace $Q$ by $B$ in 15
17	$((C \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg C))$	replace $P$ by $C$ in 16
18	$((C \Rightarrow (A \vee (Q \vee P))) \Rightarrow (\neg(A \vee (Q \vee P)) \Rightarrow \neg C))$	replace $B$ by $(A \vee (Q \vee P))$ in 17
19	$(((((Q \vee P) \vee A) \Rightarrow (A \vee (Q \vee P))) \Rightarrow (\neg(A \vee (Q \vee P)) \Rightarrow \neg((Q \vee P) \vee A)))$	replace $C$ by $((Q \vee P) \vee A)$ in 18
20	$(\neg(A \vee (Q \vee P)) \Rightarrow \neg((Q \vee P) \vee A))$	MP with 12, 19
21	$((P \Rightarrow Q) \Rightarrow ((A \vee P) \Rightarrow (A \vee Q)))$	add axiom <b>axiom4</b>
22	$((P \Rightarrow Q) \Rightarrow ((B \vee P) \Rightarrow (B \vee Q)))$	replace $A$ by $B$ in 21
23	$((P \Rightarrow C) \Rightarrow ((B \vee P) \Rightarrow (B \vee C)))$	replace $Q$ by $C$ in 22
24	$((D \Rightarrow C) \Rightarrow ((B \vee D) \Rightarrow (B \vee C)))$	replace $P$ by $D$ in 23
25	$((D \Rightarrow C) \Rightarrow (((A \vee Q) \vee P) \vee D) \Rightarrow (((A \vee Q) \vee P) \vee C))$	replace $B$ by $((A \vee Q) \vee P)$ in 24
26	$((D \Rightarrow \neg((Q \vee P) \vee A)) \Rightarrow (((A \vee Q) \vee P) \vee D) \Rightarrow (((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A)))$	replace $C$ by $\neg((Q \vee P) \vee A)$ in 25
27	$((\neg(A \vee (Q \vee P)) \Rightarrow \neg((Q \vee P) \vee A)) \Rightarrow (((A \vee Q) \vee P) \vee \neg(A \vee (Q \vee P))) \Rightarrow (((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A)))$	replace $D$ by $\neg(A \vee (Q \vee P))$ in 26
28	$(((((A \vee Q) \vee P) \vee \neg(A \vee (Q \vee P))) \Rightarrow (((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A)))$	MP with 20, 27
29	$((P \vee Q) \Rightarrow (Q \vee P))$	add axiom <b>axiom3</b>
30	$((P \vee B) \Rightarrow (B \vee P))$	replace $Q$ by $B$ in 29
31	$((C \vee B) \Rightarrow (B \vee C))$	replace $P$ by $C$ in 30
32	$((C \vee \neg((Q \vee P) \vee A)) \Rightarrow (\neg((Q \vee P) \vee A) \vee C))$	replace $B$ by $\neg((Q \vee P) \vee A)$ in 31
33	$(((((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A)) \Rightarrow (\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P)))$	replace $C$ by $((A \vee Q) \vee P)$ in 32
34	$((P \Rightarrow Q) \Rightarrow ((A \Rightarrow P) \Rightarrow (A \Rightarrow Q)))$	add sentence <b>hilb1</b>
35	$((P \Rightarrow Q) \Rightarrow ((B \Rightarrow P) \Rightarrow (B \Rightarrow Q)))$	replace $A$ by $B$ in 34
36	$((P \Rightarrow C) \Rightarrow ((B \Rightarrow P) \Rightarrow (B \Rightarrow C)))$	replace $Q$ by $C$ in 35
37	$((D \Rightarrow C) \Rightarrow ((B \Rightarrow D) \Rightarrow (B \Rightarrow C)))$	replace $P$ by $D$ in 36
38	$((D \Rightarrow C) \Rightarrow (((((A \vee Q) \vee P) \vee \neg(A \vee (Q \vee P))) \Rightarrow D) \Rightarrow (((A \vee Q) \vee P) \vee \neg(A \vee (Q \vee P))) \Rightarrow C)))$	replace $B$ by $((A \vee Q) \vee P) \vee \neg(A \vee (Q \vee P))$ in 37
39	$((D \Rightarrow (\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P))) \Rightarrow (((((A \vee Q) \vee P) \vee \neg(A \vee (Q \vee P))) \Rightarrow D) \Rightarrow (((A \vee Q) \vee P) \vee \neg(A \vee (Q \vee P))) \Rightarrow \neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P))))$	replace $C$ by $(\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P))$ in 38
40	$(((((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A)) \Rightarrow (\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P))) \Rightarrow (((((A \vee Q) \vee P) \vee \neg(A \vee (Q \vee P))) \Rightarrow ((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A))) \Rightarrow (((A \vee Q) \vee P) \vee \neg(A \vee (Q \vee P))) \Rightarrow \neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P))))$	replace $D$ by $((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A)$ in 39

41	$(((((A \vee Q) \vee P) \vee \neg(A \vee (Q \vee P))) \Rightarrow ((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A))) \Rightarrow (((A \vee Q) \vee P) \vee \neg(A \vee (Q \vee P))) \Rightarrow (\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P)))$	MP with 33, 40
42	$(((((A \vee Q) \vee P) \vee \neg(A \vee (Q \vee P))) \Rightarrow (\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P)))$	MP with 28, 41
43	$((C \vee ((A \vee Q) \vee P)) \Rightarrow (((A \vee Q) \vee P) \vee C))$	replace $B$ by $((A \vee Q) \vee P)$ in 31
44	$((\neg(A \vee (Q \vee P)) \vee ((A \vee Q) \vee P)) \Rightarrow (((A \vee Q) \vee P) \vee \neg(A \vee (Q \vee P))))$	replace $C$ by $\neg(A \vee (Q \vee P))$ in 43
45	$((D \Rightarrow C) \Rightarrow (((\neg(A \vee (Q \vee P)) \vee ((A \vee Q) \vee P)) \Rightarrow D) \Rightarrow ((\neg(A \vee (Q \vee P)) \vee ((A \vee Q) \vee P)) \Rightarrow C)))$	replace $B$ by $(\neg(A \vee (Q \vee P)) \vee ((A \vee Q) \vee P))$ in 37
46	$((D \Rightarrow (\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P))) \Rightarrow (((\neg(A \vee (Q \vee P)) \vee ((A \vee Q) \vee P)) \Rightarrow D) \Rightarrow ((\neg(A \vee (Q \vee P)) \vee ((A \vee Q) \vee P)) \Rightarrow (\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P))))$	replace $C$ by $(\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P))$ in 45
47	$(((((A \vee Q) \vee P) \vee \neg(A \vee (Q \vee P))) \Rightarrow (\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P))) \Rightarrow (((\neg(A \vee (Q \vee P)) \vee ((A \vee Q) \vee P)) \Rightarrow ((\neg(A \vee (Q \vee P)) \vee ((A \vee Q) \vee P)) \vee \neg(A \vee (Q \vee P)))) \Rightarrow ((\neg(A \vee (Q \vee P)) \vee ((A \vee Q) \vee P)) \Rightarrow (\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P))))$	replace $D$ by $((\neg(A \vee (Q \vee P)) \vee ((A \vee Q) \vee P)) \vee \neg(A \vee (Q \vee P)))$ in 46
48	$((\neg(A \vee (Q \vee P)) \vee ((A \vee Q) \vee P)) \Rightarrow (((A \vee Q) \vee P) \vee \neg(A \vee (Q \vee P)))) \Rightarrow ((\neg(A \vee (Q \vee P)) \vee ((A \vee Q) \vee P)) \Rightarrow (\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P)))$	MP with 42, 47
49	$((\neg(A \vee (Q \vee P)) \vee ((A \vee Q) \vee P)) \Rightarrow (\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P)))$	MP with 44, 48
50	$((P \Rightarrow B) \Rightarrow (\neg P \vee B))$	replace $Q$ by $B$ in 13
51	$((C \Rightarrow B) \Rightarrow (\neg C \vee B))$	replace $P$ by $C$ in 50
52	$((C \Rightarrow ((A \vee Q) \vee P)) \Rightarrow (\neg C \vee ((A \vee Q) \vee P)))$	replace $B$ by $((A \vee Q) \vee P)$ in 51
53	$((A \vee (Q \vee P)) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow (\neg(A \vee (Q \vee P)) \vee ((A \vee Q) \vee P))$	replace $C$ by $(A \vee (Q \vee P))$ in 52
54	$((D \Rightarrow C) \Rightarrow (((A \vee (Q \vee P)) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow D) \Rightarrow (((A \vee (Q \vee P)) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow C)))$	replace $B$ by $((A \vee (Q \vee P)) \Rightarrow ((A \vee Q) \vee P))$ in 37
55	$((D \Rightarrow (\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P))) \Rightarrow (((A \vee (Q \vee P)) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow D) \Rightarrow (((A \vee (Q \vee P)) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow (\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P))))$	replace $C$ by $(\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P))$ in 54
56	$((\neg(A \vee (Q \vee P)) \vee ((A \vee Q) \vee P)) \Rightarrow (\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P))) \Rightarrow (((A \vee (Q \vee P)) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow (\neg(A \vee (Q \vee P)) \vee ((A \vee Q) \vee P))) \Rightarrow (((A \vee (Q \vee P)) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow (\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P))))$	replace $D$ by $(\neg(A \vee (Q \vee P)) \vee ((A \vee Q) \vee P))$ in 55
57	$((A \vee (Q \vee P)) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow (\neg(A \vee (Q \vee P)) \vee ((A \vee Q) \vee P)) \Rightarrow (((A \vee (Q \vee P)) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow (\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P)))$	MP with 49, 56
58	$((A \vee (Q \vee P)) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow (\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P))$	MP with 53, 57
59	$((\neg P \vee B) \Rightarrow (P \Rightarrow B))$	replace $Q$ by $B$ in 14
60	$((\neg C \vee B) \Rightarrow (C \Rightarrow B))$	replace $P$ by $C$ in 59
61	$((\neg C \vee ((A \vee Q) \vee P)) \Rightarrow (C \Rightarrow ((A \vee Q) \vee P)))$	replace $B$ by $((A \vee Q) \vee P)$ in 60

62	$((\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P)) \Rightarrow (((Q \vee P) \vee A) \Rightarrow ((A \vee Q) \vee P)))$	replace $C$ by $((Q \vee P) \vee A)$ in 61
63	$((D \Rightarrow (((Q \vee P) \vee A) \Rightarrow ((A \vee Q) \vee P))) \Rightarrow (((A \vee (Q \vee P)) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow D) \Rightarrow (((A \vee (Q \vee P)) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow (((Q \vee P) \vee A) \Rightarrow ((A \vee Q) \vee P))))$	replace $C$ by $((Q \vee P) \vee A) \Rightarrow ((A \vee Q) \vee P)$ in 54
64	$((\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P)) \Rightarrow (((Q \vee P) \vee A) \Rightarrow ((A \vee Q) \vee P))) \Rightarrow (((A \vee (Q \vee P)) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow (\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P))) \Rightarrow (((A \vee (Q \vee P)) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow (((Q \vee P) \vee A) \Rightarrow ((A \vee Q) \vee P))))$	replace $D$ by $(\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P))$ in 63
65	$((((A \vee (Q \vee P)) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow (\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P))) \Rightarrow (((A \vee (Q \vee P)) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow (((Q \vee P) \vee A) \Rightarrow ((A \vee Q) \vee P))))$	MP with 62, 64
66	$((A \vee (Q \vee P)) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow (((Q \vee P) \vee A) \Rightarrow ((A \vee Q) \vee P))$	MP with 58, 65
67	$((Q \vee P) \vee A) \Rightarrow (A \vee Q) \vee P$	MP with 7, 66
68	$(D \Rightarrow C) \Rightarrow ((A \vee D) \Rightarrow (A \vee C))$	replace $B$ by $A$ in 24
69	$(D \Rightarrow (Q \vee P)) \Rightarrow ((A \vee D) \Rightarrow (A \vee (Q \vee P)))$	replace $C$ by $(Q \vee P)$ in 68
70	$((P \vee Q) \Rightarrow (Q \vee P)) \Rightarrow ((A \vee (P \vee Q)) \Rightarrow (A \vee (Q \vee P)))$	replace $D$ by $(P \vee Q)$ in 69
71	$(A \vee (P \vee Q)) \Rightarrow (A \vee (Q \vee P))$	MP with 29, 70
72	$(C \vee (Q \vee P)) \Rightarrow ((Q \vee P) \vee C)$	replace $B$ by $(Q \vee P)$ in 31
73	$(A \vee (Q \vee P)) \Rightarrow ((Q \vee P) \vee A)$	replace $C$ by $A$ in 72
74	$(D \Rightarrow C) \Rightarrow (((A \vee (P \vee Q)) \Rightarrow D) \Rightarrow ((A \vee (P \vee Q)) \Rightarrow C))$	replace $B$ by $(A \vee (P \vee Q))$ in 37
75	$(D \Rightarrow ((Q \vee P) \vee A)) \Rightarrow (((A \vee (P \vee Q)) \Rightarrow D) \Rightarrow ((A \vee (P \vee Q)) \Rightarrow ((Q \vee P) \vee A)))$	replace $C$ by $((Q \vee P) \vee A)$ in 74
76	$((A \vee (Q \vee P)) \Rightarrow ((Q \vee P) \vee A)) \Rightarrow (((A \vee (P \vee Q)) \Rightarrow (A \vee (Q \vee P))) \Rightarrow ((A \vee (P \vee Q)) \Rightarrow ((Q \vee P) \vee A)))$	replace $D$ by $(A \vee (Q \vee P))$ in 75
77	$((A \vee (P \vee Q)) \Rightarrow (A \vee (Q \vee P))) \Rightarrow ((A \vee (P \vee Q)) \Rightarrow ((Q \vee P) \vee A))$	MP with 73, 76
78	$(A \vee (P \vee Q)) \Rightarrow ((Q \vee P) \vee A)$	MP with 71, 77
79	$(C \vee A) \Rightarrow (A \vee C)$	replace $B$ by $A$ in 31
80	$((P \vee Q) \vee A) \Rightarrow (A \vee (P \vee Q))$	replace $C$ by $(P \vee Q)$ in 79
81	$(D \Rightarrow C) \Rightarrow (((P \vee Q) \vee A) \Rightarrow D) \Rightarrow (((P \vee Q) \vee A) \Rightarrow C)$	replace $B$ by $((P \vee Q) \vee A)$ in 37
82	$(D \Rightarrow ((Q \vee P) \vee A)) \Rightarrow (((P \vee Q) \vee A) \Rightarrow D) \Rightarrow (((P \vee Q) \vee A) \Rightarrow ((Q \vee P) \vee A))$	replace $C$ by $((Q \vee P) \vee A)$ in 81
83	$((A \vee (P \vee Q)) \Rightarrow ((Q \vee P) \vee A)) \Rightarrow (((P \vee Q) \vee A) \Rightarrow (A \vee (P \vee Q))) \Rightarrow (((P \vee Q) \vee A) \Rightarrow ((Q \vee P) \vee A))$	replace $D$ by $(A \vee (P \vee Q))$ in 82
84	$((P \vee Q) \vee A) \Rightarrow (A \vee (P \vee Q)) \Rightarrow (((P \vee Q) \vee A) \Rightarrow ((Q \vee P) \vee A))$	MP with 78, 83
85	$((P \vee Q) \vee A) \Rightarrow ((Q \vee P) \vee A)$	MP with 80, 84
86	$(C \Rightarrow ((Q \vee P) \vee A)) \Rightarrow (\neg((Q \vee P) \vee A) \Rightarrow \neg C)$	replace $B$ by $((Q \vee P) \vee A)$ in 17
87	$((P \vee Q) \vee A) \Rightarrow ((Q \vee P) \vee A) \Rightarrow (\neg((Q \vee P) \vee A) \Rightarrow \neg((P \vee Q) \vee A))$	replace $C$ by $((P \vee Q) \vee A)$ in 86
88	$(\neg((Q \vee P) \vee A) \Rightarrow \neg((P \vee Q) \vee A))$	MP with 85, 87

89	$((D \Rightarrow \neg((P \vee Q) \vee A)) \Rightarrow (((A \vee Q) \vee P) \vee D) \Rightarrow$ $((A \vee Q) \vee P) \vee \neg((P \vee Q) \vee A)))$	replace $C$ by $\neg((P \vee Q) \vee A)$ in 25
90	$((\neg((Q \vee P) \vee A) \Rightarrow \neg((P \vee Q) \vee A)) \Rightarrow (((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A)) \Rightarrow$ $((A \vee Q) \vee P) \vee \neg((P \vee Q) \vee A)))$	replace $D$ by $\neg((Q \vee P) \vee A)$ in 89
91	$((((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A)) \Rightarrow (((A \vee Q) \vee P) \vee \neg((P \vee Q) \vee A)))$	MP with 88, 90
92	$((C \vee \neg((P \vee Q) \vee A)) \Rightarrow (\neg((P \vee Q) \vee A) \vee C))$	replace $B$ by $\neg((P \vee Q) \vee A)$ in 31
93	$((((A \vee Q) \vee P) \vee \neg((P \vee Q) \vee A)) \Rightarrow (\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P)))$	replace $C$ by $((A \vee Q) \vee P)$ in 92
94	$((D \Rightarrow C) \Rightarrow (((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A)) \Rightarrow D) \Rightarrow$ $((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A) \Rightarrow C))$	replace $B$ by $((A \vee Q) \vee P) \vee$ $\neg((Q \vee P) \vee A)$ in 37
95	$((D \Rightarrow (\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))) \Rightarrow$ $((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A) \Rightarrow D) \Rightarrow$ $((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A) \Rightarrow (\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P)))$	replace $C$ by $(\neg((P \vee Q) \vee A) \vee$ $((A \vee Q) \vee P))$ in 94
96	$((((A \vee Q) \vee P) \vee \neg((P \vee Q) \vee A)) \Rightarrow (\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))) \Rightarrow$ $((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A) \Rightarrow ((A \vee Q) \vee P) \vee \neg((P \vee Q) \vee A)) \Rightarrow$ $((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A) \Rightarrow$ $(\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))$	replace $D$ by $((A \vee Q) \vee P) \vee$ $\neg((P \vee Q) \vee A)$ in 95
97	$((((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A)) \Rightarrow ((A \vee Q) \vee P) \vee \neg((P \vee Q) \vee A)) \Rightarrow$ $((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A) \Rightarrow (\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))$	MP with 93, 96
98	$((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A) \Rightarrow (\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))$	MP with 91, 97
99	$((\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P)) \Rightarrow ((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A))$	replace $C$ by $\neg((Q \vee P) \vee A)$ in 43
100	$((D \Rightarrow C) \Rightarrow ((\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P)) \Rightarrow D) \Rightarrow$ $(\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P)) \Rightarrow C))$	replace $B$ by $(\neg((Q \vee P) \vee A) \vee$ $((A \vee Q) \vee P))$ in 37
101	$((D \Rightarrow (\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))) \Rightarrow$ $((\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P)) \Rightarrow D) \Rightarrow$ $(\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P)) \Rightarrow (\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))$	replace $C$ by $(\neg((P \vee Q) \vee A) \vee$ $((A \vee Q) \vee P))$ in 100
102	$((((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A)) \Rightarrow (\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))) \Rightarrow$ $((\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P)) \Rightarrow ((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A)) \Rightarrow$ $((\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P)) \Rightarrow$ $(\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))$	replace $D$ by $((A \vee Q) \vee P) \vee$ $\neg((Q \vee P) \vee A)$ in 101
103	$((\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P)) \Rightarrow ((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A)) \Rightarrow$ $(\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P)) \Rightarrow (\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))$	MP with 98, 102
104	$((\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P)) \Rightarrow (\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P)))$	MP with 99, 103
105	$((((Q \vee P) \vee A) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow (\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P)))$	replace $C$ by $((Q \vee P) \vee A)$ in 52
106	$((D \Rightarrow C) \Rightarrow (((Q \vee P) \vee A) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow D) \Rightarrow$ $((Q \vee P) \vee A) \Rightarrow ((A \vee Q) \vee P) \Rightarrow C))$	replace $B$ by $((Q \vee P) \vee A) \Rightarrow$ $((A \vee Q) \vee P)$ in 37
107	$((D \Rightarrow (\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))) \Rightarrow$ $((Q \vee P) \vee A) \Rightarrow ((A \vee Q) \vee P) \Rightarrow D) \Rightarrow$ $((Q \vee P) \vee A) \Rightarrow ((A \vee Q) \vee P) \Rightarrow (\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))$	replace $C$ by $(\neg((P \vee Q) \vee A) \vee$ $((A \vee Q) \vee P))$ in 106

108	$((\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P)) \Rightarrow \neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P)) \Rightarrow (((((Q \vee P) \vee A) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow \neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P))) \Rightarrow (((((Q \vee P) \vee A) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow \neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))))$	replace $D$ by $(\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P))$ in 107
109	$(((((Q \vee P) \vee A) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow \neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P)) \Rightarrow (((((Q \vee P) \vee A) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow \neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))))$	MP with 104, 108
110	$(((((Q \vee P) \vee A) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow \neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P)) \Rightarrow (((((Q \vee P) \vee A) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow \neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))))$	MP with 105, 109
111	$((\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P)) \Rightarrow (((P \vee Q) \vee A) \Rightarrow ((A \vee Q) \vee P)))$	replace $C$ by $((P \vee Q) \vee A)$ in 61
112	$((D \Rightarrow (((P \vee Q) \vee A) \Rightarrow ((A \vee Q) \vee P))) \Rightarrow (((((Q \vee P) \vee A) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow D) \Rightarrow (((((Q \vee P) \vee A) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow ((P \vee Q) \vee A) \Rightarrow ((A \vee Q) \vee P))))$	replace $C$ by $((P \vee Q) \vee A)$ in 106
113	$((\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P)) \Rightarrow (((P \vee Q) \vee A) \Rightarrow ((A \vee Q) \vee P))) \Rightarrow ((((((Q \vee P) \vee A) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow \neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))) \Rightarrow (((((Q \vee P) \vee A) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow ((P \vee Q) \vee A) \Rightarrow ((A \vee Q) \vee P))))$	replace $D$ by $(\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))$ in 112
114	$(((((Q \vee P) \vee A) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow \neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P)) \Rightarrow (((((Q \vee P) \vee A) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow ((P \vee Q) \vee A) \Rightarrow ((A \vee Q) \vee P))))$	MP with 111, 113
115	$(((((Q \vee P) \vee A) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow ((P \vee Q) \vee A) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow (((((Q \vee P) \vee A) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow ((P \vee Q) \vee A) \Rightarrow ((A \vee Q) \vee P))))$	MP with 110, 114
116	$((P \vee Q) \vee A) \Rightarrow ((A \vee Q) \vee P)$	MP with 67, 115
117	$(C \vee P) \Rightarrow (P \vee C)$	replace $B$ by $P$ in 31
118	$((A \vee Q) \vee P) \Rightarrow (P \vee (A \vee Q))$	replace $C$ by $(A \vee Q)$ in 117
119	$((D \Rightarrow C) \Rightarrow (\neg((P \vee Q) \vee A) \vee D) \Rightarrow \neg((P \vee Q) \vee A) \vee C))$	replace $B$ by $(\neg((P \vee Q) \vee A)$ in 24
120	$((D \Rightarrow (P \vee (A \vee Q))) \Rightarrow (\neg((P \vee Q) \vee A) \vee D) \Rightarrow \neg((P \vee Q) \vee A) \vee (P \vee (A \vee Q))))$	replace $C$ by $(P \vee (A \vee Q))$ in 119
121	$((((A \vee Q) \vee P) \Rightarrow (P \vee (A \vee Q))) \Rightarrow (\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P)) \Rightarrow \neg((P \vee Q) \vee A) \vee (P \vee (A \vee Q))))$	replace $D$ by $((A \vee Q) \vee P)$ in 120
122	$((\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P)) \Rightarrow \neg((P \vee Q) \vee A) \vee (P \vee (A \vee Q)))$	MP with 118, 121
123	$((((P \vee Q) \vee A) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow \neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))$	replace $C$ by $((P \vee Q) \vee A)$ in 52
124	$((D \Rightarrow C) \Rightarrow (((((P \vee Q) \vee A) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow D) \Rightarrow (((((P \vee Q) \vee A) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow C))))$	replace $B$ by $((P \vee Q) \vee A)$ in 37
125	$((D \Rightarrow \neg((P \vee Q) \vee A) \vee (P \vee (A \vee Q))) \Rightarrow ((((((P \vee Q) \vee A) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow D) \Rightarrow (((((P \vee Q) \vee A) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow \neg((P \vee Q) \vee A) \vee (P \vee (A \vee Q))))$	replace $C$ by $(\neg((P \vee Q) \vee A) \vee (P \vee (A \vee Q))$ in 124
126	$((\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P)) \Rightarrow \neg((P \vee Q) \vee A) \vee (P \vee (A \vee Q))) \Rightarrow ((((((P \vee Q) \vee A) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow \neg((P \vee Q) \vee A) \vee (P \vee (A \vee Q)))) \Rightarrow (((((P \vee Q) \vee A) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow \neg((P \vee Q) \vee A) \vee (P \vee (A \vee Q))))$	replace $D$ by $(\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))$ in 125
127	$(((((P \vee Q) \vee A) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow \neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P)) \Rightarrow (((((P \vee Q) \vee A) \Rightarrow ((A \vee Q) \vee P)) \Rightarrow \neg((P \vee Q) \vee A) \vee (P \vee (A \vee Q))))$	MP with 122, 126



151	$(((P \vee Q) \vee A) \Rightarrow (P \vee (A \vee Q))) \Rightarrow (\neg((P \vee Q) \vee A) \vee (P \vee (Q \vee A)))$	MP with 146, 150
152	$((\neg C \vee (P \vee (Q \vee A))) \Rightarrow (C \Rightarrow (P \vee (Q \vee A))))$	replace $B$ by $(P \vee (Q \vee A))$ in 60
153	$((\neg((P \vee Q) \vee A) \vee (P \vee (Q \vee A))) \Rightarrow (((P \vee Q) \vee A) \Rightarrow (P \vee (Q \vee A))))$	replace $C$ by $((P \vee Q) \vee A)$ in 152
154	$((D \Rightarrow (((P \vee Q) \vee A) \Rightarrow (P \vee (Q \vee A)))) \Rightarrow (((((P \vee Q) \vee A) \Rightarrow (P \vee (A \vee Q))) \Rightarrow D) \Rightarrow (((((P \vee Q) \vee A) \Rightarrow (P \vee (A \vee Q))) \Rightarrow (((P \vee Q) \vee A) \Rightarrow (P \vee (Q \vee A)))))))$	replace $C$ by $((P \vee Q) \vee A) \Rightarrow (P \vee (Q \vee A))$ in 147
155	$(((\neg((P \vee Q) \vee A) \vee (P \vee (Q \vee A))) \Rightarrow (((P \vee Q) \vee A) \Rightarrow (P \vee (Q \vee A)))) \Rightarrow ((((((P \vee Q) \vee A) \Rightarrow (P \vee (A \vee Q))) \Rightarrow (\neg((P \vee Q) \vee A) \vee (P \vee (Q \vee A)))) \Rightarrow (((((P \vee Q) \vee A) \Rightarrow (P \vee (A \vee Q))) \Rightarrow (((P \vee Q) \vee A) \Rightarrow (P \vee (Q \vee A)))))))$	replace $D$ by $(\neg((P \vee Q) \vee A) \vee (P \vee (Q \vee A)))$ in 154
156	$((((((P \vee Q) \vee A) \Rightarrow (P \vee (A \vee Q))) \Rightarrow (\neg((P \vee Q) \vee A) \vee (P \vee (Q \vee A)))) \Rightarrow (((((P \vee Q) \vee A) \Rightarrow (P \vee (A \vee Q))) \Rightarrow (((P \vee Q) \vee A) \Rightarrow (P \vee (Q \vee A))))))$	MP with 153, 155
157	$((((((P \vee Q) \vee A) \Rightarrow (P \vee (A \vee Q))) \Rightarrow (((P \vee Q) \vee A) \Rightarrow (P \vee (Q \vee A)))) \Rightarrow (((P \vee Q) \vee A) \Rightarrow (P \vee (Q \vee A))))$	MP with 151, 156
158	$(((P \vee Q) \vee A) \Rightarrow (P \vee (Q \vee A)))$	MP with 135, 157

□

Another consequence from hilb13 is the exchange of preconditions:

**Theorem 0.22.**

$$((P \Rightarrow (Q \Rightarrow A)) \Rightarrow (Q \Rightarrow (P \Rightarrow A)))$$

*Proof.*

1	$((P \vee (Q \vee A)) \Rightarrow (Q \vee (P \vee A)))$	add sentence hilb13
2	$((P \vee (Q \vee B)) \Rightarrow (Q \vee (P \vee B)))$	replace $A$ by $B$ in 1
3	$((P \vee (C \vee B)) \Rightarrow (C \vee (P \vee B)))$	replace $Q$ by $C$ in 2
4	$((D \vee (C \vee B)) \Rightarrow (C \vee (D \vee B)))$	replace $P$ by $D$ in 3
5	$((D \vee (C \vee A)) \Rightarrow (C \vee (D \vee A)))$	replace $B$ by $A$ in 4
6	$((D \vee (\neg Q \vee A)) \Rightarrow (\neg Q \vee (D \vee A)))$	replace $C$ by $\neg Q$ in 5
7	$((\neg P \vee (\neg Q \vee A)) \Rightarrow (\neg Q \vee (\neg P \vee A)))$	replace $D$ by $\neg P$ in 6
8	$((P \Rightarrow (\neg Q \vee A)) \Rightarrow (\neg Q \vee (\neg P \vee A)))$	reverse abbreviation impl in 7 at occurrence 1
9	$((P \Rightarrow (Q \Rightarrow A)) \Rightarrow (\neg Q \vee (\neg P \vee A)))$	reverse abbreviation impl in 8 at occurrence 1
10	$((P \Rightarrow (Q \Rightarrow A)) \Rightarrow (Q \Rightarrow (\neg P \vee A)))$	reverse abbreviation impl in 9 at occurrence 1
11	$((P \Rightarrow (Q \Rightarrow A)) \Rightarrow (Q \Rightarrow (P \Rightarrow A)))$	reverse abbreviation impl in 10 at occurrence 1

□

An analogous form for ??:

**Theorem 0.23.**

$$((Q \Rightarrow (P \Rightarrow A)) \Rightarrow (P \Rightarrow (Q \Rightarrow A)))$$

*Proof.*

1	$((P \Rightarrow (Q \Rightarrow A)) \Rightarrow (Q \Rightarrow (P \Rightarrow A)))$	add sentence <b>hilb16</b>
2	$((P \Rightarrow (Q \Rightarrow B)) \Rightarrow (Q \Rightarrow (P \Rightarrow B)))$	replace $A$ by $B$ in <b>1</b>
3	$((P \Rightarrow (C \Rightarrow B)) \Rightarrow (C \Rightarrow (P \Rightarrow B)))$	replace $Q$ by $C$ in <b>2</b>
4	$((D \Rightarrow (C \Rightarrow B)) \Rightarrow (C \Rightarrow (D \Rightarrow B)))$	replace $P$ by $D$ in <b>3</b>
5	$((D \Rightarrow (C \Rightarrow A)) \Rightarrow (C \Rightarrow (D \Rightarrow A)))$	replace $B$ by $A$ in <b>4</b>
6	$((D \Rightarrow (P \Rightarrow A)) \Rightarrow (P \Rightarrow (D \Rightarrow A)))$	replace $C$ by $P$ in <b>5</b>
7	$((Q \Rightarrow (P \Rightarrow A)) \Rightarrow (P \Rightarrow (Q \Rightarrow A)))$	replace $D$ by $Q$ in <b>6</b>

□

This module used by the following modules:

Name: prophilbert2  
 Version: 1.00.00  
 Rule version: 1.00.00  
 Orgin: [prophilbert2\\_1.00.00\\_1.00.00.qedeq](#)  
 pdf: [prophilbert2\\_1.00.00\\_1.00.00.pdf](#)

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